Review

Euler : $y_{n+1} = y_n + f(t_n, y_n)\,h$

Improved Euler : $y_{n+1} = y_n + \frac{1}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]h$

y_{n+1} on the right side is

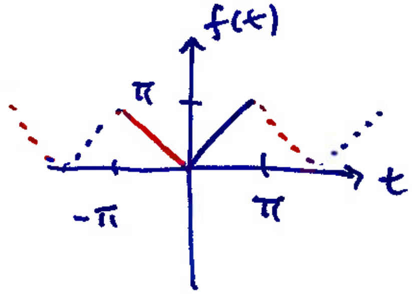
Euler $y_{n+1} = y_n + f(t_n, y_n)h$

No RK4 on exam.

Sine / cosine series

$f(t) = t \qquad 0 < t < \pi \quad$ period 2π

cosine series $\rightarrow$ add even extensions



cosine series has no sine terms

$$b_n = 0 \quad \text{for all } n$$

$$a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

here, $L = \pi$

$$a_0 = \underbrace{\frac{2}{\pi}\underbrace{\int_0^{\pi} t\, dt}_{\text{area}}}_{\text{avg. value of } f(t) \text{ when in series } \left(\frac{a_0}{2}\right)} = \frac{2}{\pi}\left[(\pi)(\pi)\left(\frac{1}{2}\right)\right] = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} t\cos(nt)\,dt \qquad u = t \qquad dv = \cos(nt)\,dt$$

$$du = dt \qquad v = \frac{1}{n}\sin(nt)$$

$$= \frac{2}{\pi}\left(\frac{t}{n}\sin(nt)\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nt)\,dt\right)$$

$$= -\frac{2}{n\pi}\left(-\frac{1}{n}\cos(nt)\Big|_0^{\pi} = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]\right.$$

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nt)$$

even · even is even

even · odd is odd

odd · odd is even

if $f(t)$ even

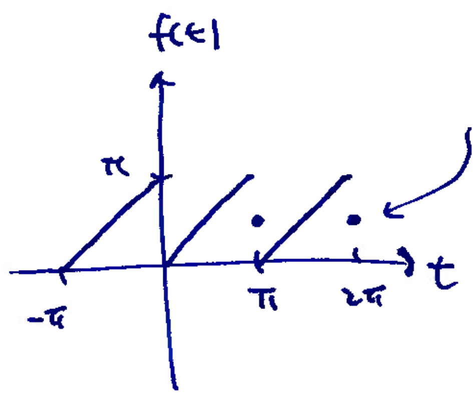
$f(t) \sin\left(\frac{n\pi t}{L}\right)$ is even · odd = odd

so $b_n \neq 0$ in general

$$f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right)$$

~~Conver~~ converges to $f(t)$ if $f(t)$ is continuous at some t

converges to $\frac{f(t^-)+f(t^+)}{2}$ if $f(t)$ is discontinuous at t



converges to $\frac{\pi}{2}$ at $n\pi$

converges to $f(t)=t$ at other t

Boundary-value problem

$$X'' + kX = F(t) \qquad X(0) = X(L) = 0$$
$$X'(0) = X'(L) = 0$$

with a given $F(t)$ on $0 < t < L$

we add even extensions if $X'(0) = X'(L) = 0$

we add odd extensions if $X(0) = X(L) = 0$

expand $F(t)$

assume $X(t)$ is the same kind of series

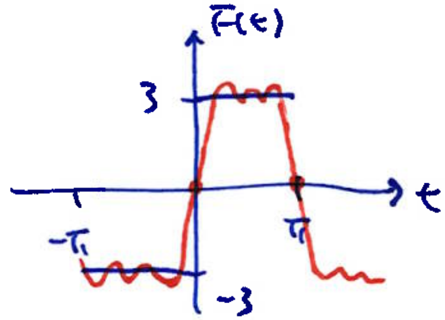
plug into eq. to find coefficients

for example, $X'' + 5X = F(t) \qquad X(0) = X(\pi) = 0$

$F(t) = 3 \quad 0 < t < \pi$

$X(0) = X(\pi) = 0$ (positions fixed)

we expand $F(t) = 3$ as a sine series w/ period 2π



* $F(0) = F(\pi) = 0$ (meets BC's)

expand $F(t) = 3 \quad 0 < t < \pi \quad (L = \pi)$ as sine series

$a_n = 0$ for all n

$$b_n = \frac{2}{\pi}\int_0^{\pi} 3 \cdot \sin(nt)\,dt = \cdots$$

$$= \frac{6\left[1-(-1)^n\right]}{n\pi}$$

$$F(t) = \sum_{n=1}^{\infty} \frac{6\left[1-(-1)^n\right]}{n\pi} \sin(nt)$$

we assume solution of the form $x(t) = \sum_{n=1}^{\infty} B_n \sin(nt)$

sub into $x'' + 5x = F(t) = \sum_{n=1}^{\infty} \frac{6\left[1-(-1)^n\right]}{n\pi} \sin(nt)$

$$x' = \sum_{n=1}^{\infty} n B_n \cos(nt)$$

$$x'' = \sum_{n=1}^{\infty} -n^2 B_n \sin(nt)$$

$$\sum_{n=1}^{\infty} -n^2 B_n \underline{\sin(nt)} + \sum_{n=1}^{\infty} 5 B_n \underline{\sin(nt)} = \sum_{n=1}^{\infty} \frac{6[1-(-1)^n]}{n\pi} \underline{\sin(nt)}$$

for each n, $-n^2 B_n + 5B_n = \dfrac{6[1-(-1)^n]}{n\pi}$

$$B_n = \frac{6[1-(-1)^n]}{n\pi(5-n^2)}$$

so, $x(t) = \displaystyle\sum_{n=1}^{\infty} \frac{6[1-(-1)^n]}{n\pi(5-n^2)} \sin(nt)$

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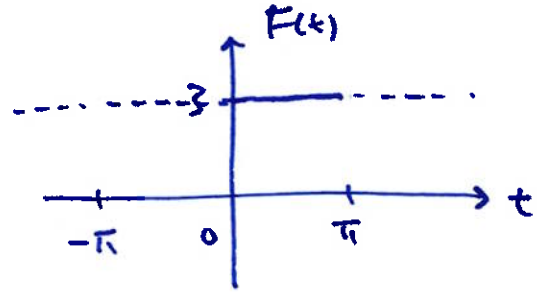
if same eq. but $x'(0) = x'(\pi) = 0$

$$F(t) = 3$$

then even extension



$b_n = 0, \quad a_0 = 6, \quad a_n = 0 \quad n \geq 1$

still assume $x(t)$ is a cosine series

$$x(t) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos(nt)$$

Fourier Series and Differential Equations Reference

Term	Formula	Period
$f(x)$	$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$	$2L$
a_n	$\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx$	$2L$
b_n	$\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$	$2L$
Even Extension	$b_n = 0, \quad a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx$	$2L$
Odd Extension	$a_n = 0, \quad b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$	$2L$

Trigonometric Identities

- $\sin(A)\cos(B) = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$
- $\cos(A)\cos(B) = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$
- $\cos(n\pi) = (-1)^n, \quad \sin(n\pi) = 0$

Integration

- $\int u\, dv = uv - \int v\, du$

Common Trigonometric Integrals (for integer m, n)

- $\int_0^{\pi} \sin(nx)\cos(mx)\, dx = \begin{cases} \frac{2n}{n^2-m^2} & \text{if } n-m \text{ is odd} \\ 0 & \text{if } n-m \text{ is even} \end{cases}$
- $\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} L & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$
- $\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} L & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$
- $\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = 0 \quad \text{for all } n, m$

Standard Form Integrals

- $\int x\cos(ax)\, dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C$
- $\int x\sin(ax)\, dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$